

A COHOMOLOGICAL SPLITTING CRITERION FOR RANK 2 VECTOR BUNDLES ON HIRZEBRUCH SURFACES

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ABSTRACT. In this note, we give a cohomological characterization of all rank 2 split vector bundles on Hirzebruch surfaces.

1. INTRODUCTION

Throughout this paper we work over an algebraic closed field k . A vector bundle on a smooth projective variety is called *split* if it is decomposed into a direct sum of line bundles. Recently, Fulger and Marchitan ([2]) obtained a cohomological characterization of some rank 2 split vector bundles on Hirzebruch surfaces over the complex number field, by using Buchdahl's Beilinson type spectral sequence ([1]). However, it seems difficult to apply their argument to general cases. The purpose of this paper is to give a simple characterization of all rank 2 split vector bundles on Hirzebruch surfaces by cohomological informations in arbitrary characteristic.

Theorem 1. *Let \mathcal{E} be a rank 2 vector bundle on a Hirzebruch surface $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. Let \mathcal{F} be a split rank 2 vector bundle on \mathbb{F}_n . If $\dim_k H^i(\mathcal{E} \otimes \mathcal{L}) = \dim_k H^i(\mathcal{F} \otimes \mathcal{L})$ for any $0 \leq i \leq 2$ and any line bundle \mathcal{L} on \mathbb{F}_n , then \mathcal{E} is isomorphic to \mathcal{F} .*

It seems that our assumption of Theorem 1 is stronger than the one of Theorem in [2]. However, if we know the Chern classes of \mathcal{E} , we need only a few assumptions (see Lemma 2). In the cases treated in [2], the assumption of Lemma 2 is essentially equivalent to the one of the Theorem of [2].

NOTATION

For a smooth projective variety X and a vector bundle \mathcal{E} on X , let $\mathbb{P}_X(\mathcal{E})$ be the projectivization of \mathcal{E} in the sense of Grothendieck.

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We denote $\dim_k H^i(\mathcal{E})$ by $h^i(\mathcal{E})$. We denote the Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ by \mathbb{F}_n , the natural projection $\mathbb{F}_n \rightarrow \mathbb{P}^1$ by π , the minimal section on \mathbb{F}_n by σ (i.e. $\sigma \cong \mathbb{P}^1$, $\sigma^2 = -n$) and a fiber of π by f . It is well-known that $\text{Pic}(\mathbb{F}_n) = \mathbb{Z}\sigma \oplus \mathbb{Z}f$.

2. PROOF OF THEOREM 1

To give the proof of Theorem 1, we show the following lemma.

Lemma 2. *Let \mathcal{E} be a rank 2 vector bundle on \mathbb{F}_n . Assume that $h^0(\mathcal{E}(-\sigma)) = h^0(\mathcal{E}(-f)) = 0$, $h^0(\mathcal{E}) \geq 1$ and $c_2(\mathcal{E}) = 0$.*

- (1) *Assume that $c_1(\mathcal{E}) = -a\sigma - bf$, where a and b are nonnegative integers, and that $h^0(\mathcal{E}(a\sigma + bf)) \geq 1 + h^0(\mathcal{O}_{\mathbb{F}_n}(a\sigma + bf))$. Then \mathcal{E} is isomorphic to $\mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(-a\sigma - bf)$.*
- (2) *Assume that $c_1(\mathcal{E}) = a\sigma - bf$, where a and b are integers such that $ab > 0$, and that $h^0(\mathcal{E}(-a\sigma + bf)) \geq 1 + h^0(\mathcal{O}_{\mathbb{F}_n}(-a\sigma + bf))$. Then \mathcal{E} is isomorphic to $\mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(a\sigma - bf)$.*

Proof. Put $\mathcal{L} = \mathcal{O}_{\mathbb{F}_n}(-a\sigma - bf)$ in Case 1 and $\mathcal{L} = \mathcal{O}_{\mathbb{F}_n}(a\sigma - bf)$ in Case 2. Since $h^0(\mathcal{E}) \neq 0$, we can take a nonzero section $0 \neq s \in H^0(\mathcal{E})$. Put $Z := (s = 0)$. If s takes zero on some nonzero effective divisor $D > 0$, we have a nonzero section of $H^0(\mathcal{E}(-D))$. This is a contradiction because $h^0(\mathcal{E}(-D)) \leq \max\{h^0(\mathcal{E}(-\sigma)), h^0(\mathcal{E}(-f))\} = 0$. Therefore, Z is of codimension 2. Since $c_2(\mathcal{E}) = 0$, we have $Z = \emptyset$. Hence we obtain an exact sequence,

$$0 \rightarrow \mathcal{O}_{\mathbb{F}_n} \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0,$$

since $c_1(\mathcal{E}) = -a\sigma - bf$ in Case 1 and $c_1(\mathcal{E}) = a\sigma - bf$ in Case 2. We show that the above exact sequence splits. Consider the long exact sequence ;

$$0 \rightarrow \text{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{F}_n}) \rightarrow \text{Hom}(\mathcal{L}, \mathcal{E}) \rightarrow \text{Hom}(\mathcal{L}, \mathcal{L}) \rightarrow \text{Ext}^1(\mathcal{L}, \mathcal{O}_{\mathbb{F}_n}) \rightarrow \cdots.$$

By the assumption, we have

$$\begin{aligned} \dim_k \text{Hom}(\mathcal{L}, \mathcal{E}) &= h^0(\mathcal{E} \otimes \mathcal{L}^\vee) \\ &\geq h^0(\mathcal{L}^\vee) + h^0(\mathcal{O}_{\mathbb{F}_n}) = \dim_k \text{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{F}_n}) + \dim_k \text{Hom}(\mathcal{L}, \mathcal{L}). \end{aligned}$$

Therefore, the homomorphism $\text{Hom}(\mathcal{L}, \mathcal{E}) \rightarrow \text{Hom}(\mathcal{L}, \mathcal{L})$ is surjective. Hence \mathcal{E} is isomorphic to $\mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{L}$. \square

Remark 3. *Let \mathcal{E} be a split rank 2 vector bundle on \mathbb{F}_n . It is readily seen that there exists a line bundle \mathcal{L} on \mathbb{F}_n such that $c_1(\mathcal{E} \otimes \mathcal{L}) = \alpha\sigma + \beta f$ with $\alpha, \beta \leq 0$ or $\alpha\beta > 0$ and that $c_2(\mathcal{E} \otimes \mathcal{L}) = 0$. Therefore, by Lemma 2, we can characterize all split rank 2 vector bundles on \mathbb{F}_n .*

From now on, we give a proof of Theorem 1. We will begin with a proof of the following Claim.

Claim 4. *Under the assumptions of Theorem 1, we have $\det(\mathcal{E}) = \det(\mathcal{F})$ in $\text{Pic}(\mathbb{F}_n)$ and $\deg c_2(\mathcal{E}) = \deg c_2(\mathcal{F})$.*

Proof. Take arbitrarily a very ample divisor D on \mathbb{F}_n . We may assume that D is smooth. By the assumptions, we have $\chi(\mathcal{E}) = \chi(\mathcal{F})$ and $\chi(\mathcal{E}(-D)) = \chi(\mathcal{F}(-D))$. Therefore, we obtain $\chi(\mathcal{E}|_D) = \chi(\mathcal{F}|_D)$. By Riemann-Roch theorem on the smooth curve D , we have $c_1(\mathcal{E}) \cdot D = c_1(\mathcal{F}) \cdot D$ (cf. [3], Example 15.2.1.). Hence $c_1(\mathcal{E})$ is numerically equivalent to $c_1(\mathcal{F})$. Because \mathbb{F}_n is rational, we get $\det(\mathcal{E}) = \det(\mathcal{F})$. We also have $\deg c_2(\mathcal{E}) = \deg c_2(\mathcal{F})$ from the Riemann-Roch theorem on \mathbb{F}_n (cf. [3], Example 15.2.2.). \square

Now we conclude the proof of Theorem 1. By Remark 3, we may assume that \mathcal{F} is isomorphic to $\mathcal{O} \oplus \mathcal{L}$, where $\mathcal{L} = \mathcal{O}_{\mathbb{F}_n}(-a\sigma - bf)$ with nonnegative integers $a, b \geq 0$ or $\mathcal{L} = \mathcal{O}_{\mathbb{F}_n}(a\sigma - bf)$ with positive integers $a, b > 0$.

By Claim 4, we have $c_1(\mathcal{E}) = c_1(\mathcal{F}) = c_1(\mathcal{L})$ and $c_2(\mathcal{E}) = 0$. By the assumptions of Theorem 1, we also have $h^0(\mathcal{E}(-\sigma)) = h^0(\mathcal{F}(-\sigma)) = 0$, $h^0(\mathcal{E}(-f)) = h^0(\mathcal{F}(-f)) = 0$, $h^0(\mathcal{E}) = h^0(\mathcal{F}) \geq 1$ and $h^0(\mathcal{E} \otimes \mathcal{L}^\vee) = h^0(\mathcal{F} \otimes \mathcal{L}^\vee) = 1 + h^0(\mathcal{L}^\vee)$. Then, by Lemma 2, we obtain the result of Theorem 1.

Similar arguments of the proof of Theorem 1 imply the following theorem.

Theorem 5. *Let S be a smooth projective surface with the Picard group $\text{Pic}(S) \cong \mathbb{Z}$. Let \mathcal{E} be a rank 2 vector bundle on S . Let \mathcal{F} be a split rank 2 vector bundle on S . If $h^i(\mathcal{E} \otimes \mathcal{L}) = h^i(\mathcal{F} \otimes \mathcal{L})$ for any $0 \leq i \leq 2$ and any line bundle \mathcal{L} on S , then \mathcal{E} is isomorphic to \mathcal{F} .*

Proof. We may assume that $\mathcal{F} \cong \mathcal{O}_S \oplus \mathcal{M}$ where \mathcal{M} is a line bundle on S such that $\deg \mathcal{M} \leq 0$. In the same manner as in Claim 4, we can verify that $\det(\mathcal{E}) = \det(\mathcal{F})$ in $\text{Pic}(S)$ since $\text{Pic}(S)$ is isomorphic to \mathbb{Z} . Moreover we also have $\deg c_2(\mathcal{E}) = \deg c_2(\mathcal{F})$. Therefore, we have an exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{E} \rightarrow \mathcal{M} \rightarrow 0.$$

Moreover, we can show that $\mathcal{E} \cong \mathcal{O}_S \oplus \mathcal{M}$ in the same way as in the proof of Theorem 1. \square

Remark 6. *There are many surfaces having the Picard group $\text{Pic}(S) \cong \mathbb{Z}$. In fact, on the complex number field, it is known that a very general*

surface $S \subseteq \mathbb{P}^3$ of degree $d \geq 4$ has the Picard group $\text{Pic}(S) \cong \mathbb{Z}$. (cf. [4], Theorem)

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